LU-Decomposition of a Matrix with Entries of Different Kinds (線型計算の標準算法と実現)

AUTHOR(S):

MUROTA, KAZUO

数理解析研究所講究録
463巻 1982年 23-32

LU-Decomposition of a Matrix with Entries of Different Kinds

Kazuo MUROTA*

February 1982

Abstract

Let $\underline{F} \supset \underline{K}$ be fields and consider a matrix A over $\underline{F}$ whose entries not belonging to $\underline{K}$ are algebraically independent transcendentals over $\underline{K}$. It is shown that if det $A \in \underline{K}^*$ $(=\underline{K}-\{0\})$, the matrix A, with suitable permutations of its rows and columns, is decomposed into LU-factors with the entries of the U-factor belonging to $\underline{K}$.

---

* Department of Mathematical Engineering and Instrumentation Physics, Faculty of Engineering, University of Tokyo, Hongo, Tokyo 113, Japan

1. Introduction

Let $\underline{K}$ be a field and $\underline{F}$ ($\supset \underline{K}$) an extension field. For $S \subset \underline{F}$ we denote by $M(S)$ the set of matrices with entries belonging to S. Suppose an n by n matrix $A=(A_{ij}) \in M(\underline{F})$ is expressed as

$$A = Q + T, \qquad (1)$$

where

i) $Q \in M(\underline{K})$,

ii) non-zero entries of T are algebraically independent transcendentals over $\underline{K}$.

In the following we shall denote by T^* the set of non-zero entries of T.

As is well known, A is invertible in the ring $\underline{K}[T^*]$ of polynomials in T^* over $\underline{K}$, i.e., $A^{-1} \in M(\underline{K}[T^*])$, iff $\det A \in \underline{K}^*(=\underline{K}-\{0\})$. Here we are interested in whether we can compute A^{-1} by means of pivot operations in $\underline{K}[T^*]$; moreover, how simple we can make the LU-factors of A by applying suitable permutations to its rows and columns.

By way of illustration, we will start with an example. Let $\underline{K}=\underline{Q}$ (the field of rational numbers) and set $\underline{F}=\underline{Q}(x,y,z)$, where $\{x,y,z\}$, as a collection, is assumed to be algebraically independent over $\underline{Q}$. Matrix

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & x & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & z & 0 \end{pmatrix} \end{array},$$

is expressed as the sum of the following Q and T according to (1):

$$Q = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 \end{pmatrix}.$$

Note that det A = 2 and hence A is invertible in $\underline{Q}[x,y,z]$. The matrix A is decomposed into LU-factors in $\underline{F}$ as

$$A = L\ U,$$

with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -y & y-1 & y-1-2/x & 1 & 0 \\ -1 & 2 & 2+1/x & -(xz+1)/2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & x+1 & 1 & 1 \\ 0 & 0 & -x & -1 & 0 \\ 0 & 0 & 0 & -2/x & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is observed that some of the entries of L and U, especially some of the diagonals of U, do not belong to $\underline{K}[T^*]$.

However, after rearranging the rows and the columns of A as

$$\tilde{A} = \begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 5 & 3 & 4 & 2 \end{array} \\ \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 0 & x & 1 & 0 \\ 1 & 0 & 0 & z & 1 \end{pmatrix} \end{array},$$

we obtain the LU-decomposition

$$\tilde{A} = \tilde{L}\ \tilde{U}$$

with

$$\tilde{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -y & y-1 & 1 & 0 & 0 \\ -1 & 1 & x/2 & 1 & 0 \\ -1 & 1 & 0 & z & 1 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The LU-factors are much simpler in the sense that all the entries of $\tilde{U}$ are numbers in $\underline{K}=\underline{Q}$, i.e., $\tilde{U} \in M(\underline{K})$ and, consequently, the entries of $\tilde{L}$ are polynomials in x, y and z over $\underline{K}$ of degree at most 1.

In this paper, we establish a theorem stating to the effect that this is always the case for any matrix A which admits the expression of (1) with $\det A \in \underline{K}^*$, i.e., that it is always possible to find a permutation of rows and that of columns, through which the matrix A can be brought to the form decomposable into LU-factors with a U-factor in $M(\underline{K})$. Furthermore, it is shown how to find suitable permutations. Some implications of the theorem are also discussed.

2. The Theorem

In this section we prove the following theorem.

Theorem. Let A be a matrix of form (1). If $\det A \in \underline{K}^*$, then there exist permutation matrices P_r, P_c and LU-factors $\tilde{L} = (\tilde{L}_{ij})$, $\tilde{U} = (\tilde{U}_{ij})$:

$$P_r' \, A \, P_c = \tilde{L}\,\tilde{U}$$

such that

(i) $\tilde{L}_{ij}$ is a polynomial of degree at most 1 in non-zero entries T^* of T over $\underline{K}$ ($\tilde{L}_{ii}=1$; $\tilde{L}_{ij}=0$ for $i<j$)

and

(ii) $\tilde{U} \in M(\underline{K})$; $\tilde{U}_{ii} \in \underline{K}^*$ ($\tilde{U}_{ij}=0$ for $i>j$). ▯

To prove the theorem, the following lemma is crucial, giving a necessary and sufficient condition for a matrix of form (1) to be invertible in $\underline{K}[T^*]$. We will say that a matrix is strictly lower triangular if it is a lower triangular matrix with zero diagonals.

Lemma 1. Let A be a matrix in (1). Then $\det A \in \underline{K}^*$ iff $\det Q \neq 0$ and $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r. ▯

Proof: ["if" part] Suppose $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r. Then, since $\det Q \neq 0$ and $A = Q + T$, we have

$$\begin{aligned}\det A &= \det[(I+TQ^{-1})\,Q] \\ &= \det[I + P_r'(TQ^{-1})P_r]\,\det Q \\ &= \det Q \in \underline{K}^*.\end{aligned}$$

["only if" part] If $\det A \in \underline{K}^*$, then $\det Q = \det A \neq 0$, so that we may put $S = Q^{-1}$. Suppose, to the contrary, that $P_r'(TS)P_r$ is not strictly lower triangular for any permutation matrix P_r. Then TS has a cycle of non-zero entries, that is, there exist an integer $M \geq 1$ and a sequence of indices $i(m)$ and $j(m)$ $(m=1,\ldots,M)$ such that

$$T_{i(m-1),j(m)} \neq 0 \text{ and } S_{j(m),i(m)} \neq 0 \text{ for } m=1,\ldots,M,$$

where $i(0)=i(M)$. Choose M to be the minimal of such integers. For notational simplicity, we write $T_{i(m-1),j(m)}=t_m$ and $S_{j(m),i(m)}=s_m$.

For $k=0,1,\ldots,$ consider the expression of the $(j(1),i(1))$ entry of $S(TS)^{kM}$ in the form of the sum of products of T_{ij}'s and S_{ji}'s. Corresponding to the above cycle, it contains a term

$$s_1(s_1 s_2 \ldots s_M)^k (t_1 \ldots t_M)^k,$$

since no other similar terms of $(t_1 \ldots t_M)^k$ exist due to the minimality of M and since it cannot be cancelled out by non-similar terms by virtue of the algebraic independence of elements of T^*.

Next we formally expand A^{-1} as

$$A^{-1} = [(I+TQ^{-1})Q]^{-1}$$

$$= S - STS + STSTS - \dots .$$

Each entry of A^{-1} on the left-hand side is a polynomial in T^* over $\underline{K}$ since $\det A \in \underline{K}^*$. On the right-hand side, we first observe that each entry of the m-th term is a homogeneous polynomial in T^* of degree $m-1$. Hence, by algebraic independence of T^*, no cancellation occurs among distinct terms in this expansion.

It follows in particular that the $(j(1),i(1))$ entry of the right-hand side contains a term of arbitrarily high degree, since the non-zero term $(t_1 \dots t_M)^k$ of degree kM, stemming from $S(TS)^{kM}$ as above, cannot be cancelled out for $k=0,1,\dots$. This is a contradiction. □

We make use of the following well-known lemma, the proof of which is omitted.

Lemma 2. If $\det Q \neq 0$, then for any permutation matrix P_r, there exists a permutation matrix P_c and LU-factors M, $\tilde{U}$ such that

$$P_r' \, Q \, P_c = M \, \tilde{U},$$

where M is a lower triangular matrix with unit diagonals in $M(\underline{K})$ and $\tilde{U}$ a nonsingular upper triangular matrix in $M(\underline{K})$. □

With Lemmas 1 and 2, the Theorem is easy to establish as shown below.

Proof of Theorem: Let P_r and P_c be permutation matrices as in Lemmas 1 and 2, respectively. Then from Lemma 2 we obtain

$$\tilde{A} = P_r' \, A \, P_c$$

$$= P_r' \, (Q+T) \, P_c$$

$$= (I+P_r'(TQ^{-1})P_r) \, (P_r'QP_c)$$

$$= (I+P_r'(TQ^{-1})P_r) \, M \, \tilde{U}$$

$$= \tilde{L}\,\tilde{U},$$

where

$$\tilde{L} = (I+P_r'(TQ^{-1})P_r)\ M.$$

Since both factors of $\tilde{L}$ are lower triangular matrices with unit diagonals, $\tilde{L}$ is also a lower triangular matrix with unit diagonals and therefore $\tilde{A} = \tilde{L}\,\tilde{U}$ is actually the LU-decomposition of $\tilde{A}$. Obviously $\tilde{U}$ belongs to $M(\underline{K})$ and, consequently, the entries of $\tilde{L} = \tilde{A}\,\tilde{U}^{-1}$ are polynomials in T^* of degree at most 1. □

Remark 1. In parallel with the Theorem, it is likewise possible to find permutations through which A can be brought to a form decomposable into LU-factors in such a way that the L-, instead of U-, factor belongs to $M(\underline{K})$.

Remark 2. Consider a matrix A in $M(\underline{F})$. Then it can be written as

$$A = Q + T,$$

where $Q \in M(\underline{K})$ and $T \in M(\underline{F} \backslash \underline{K})$. In general, the non-zero entries T^* of T are not algebraically independent over $\underline{K}$ and the LU-decomposition of the above-mentioned kind does not necessarily exist even if $\det A \in \underline{K}^*$, as is the case with

$$A = \begin{pmatrix} x & 1+x \\ 1-x & -x \end{pmatrix},$$

where $\underline{K}=\underline{Q}$ and $\underline{F}=\underline{Q}(x)$.

However, it may happen that the matrix $A_0=Q+T_0$, where T_0 is obtained from T by replacing its non-zero entries by algebraically independent transcendentals, satisfies the condition $\det A_0 \in \underline{K}^*$. Then the Theorem can be applied to A_0, which, in turn, implies that A itself can be decomposed, with suitable permutations, into the LU-factors with a U-factor belonging to $M(\underline{K})$.

3. Discussions

When given a matrix A of form (1) satisfying the condition $\det A \in \underline{K}^*$, we can find the suitable permutations P_r and P_c on the basis of Lemmas 1 and 2. P_r can be determined by the zero/non-zero pattern of TQ^{-1} and P_c by pivoting operations on the matrix Q. Thus both permutations can be found with $O(n^3)$ arithmetic operations in $\underline{K}$.

Lemma 1 gives an efficient way, with $O(n^3)$ arithmetic operations in $\underline{K}$, for testing whether a matrix A of form (1) satisfies the condition $\det A \in \underline{K}^*$.

The problem dealt with in the present paper has arisen when the author was investigating the following problem of large-scale system analysis.

Let R and C be the set of row and column numbers, respectively, and $A(I,J)$ denote the submatrix of A corresponding to $I(\subset R)$ and $J(\subset C)$. For a matrix A of form (1), it is known [1] (cf. also the concept of 2-block rank in [2]) that we can find, by an efficient algorithm, two subsets $I \subset R$ and $J \subset C$ such that

$$\operatorname{rank} A = \operatorname{rank} A(I,J) + \operatorname{rank} A(R \setminus I, C \setminus J),$$

$$\operatorname{rank} A(I,J) = \operatorname{rank} Q(I,J)$$

and

$$\operatorname{rank} A(R \setminus I, C \setminus J) = \operatorname{rank} T(R \setminus I, C \setminus J),$$

where the rank is considered over $\underline{F}$. If we take I and J to be the minimal of such subsets, we have $|I| = |J|$ and

$$\det A(I,J) \in \underline{K}^*,$$

The submatrix $A(I,J)$ above meets the condition of the Theorem. This implies that a matrix A of form (1) with $\det A \neq 0$ can be decomposed, after suitable permutations P_r and P_c, into LU-factors as

$$P_r' \, A \, P_c = \tilde{L} \, \tilde{U}$$

with a lower triangular matrix

$$\tilde{L} = \begin{pmatrix} \tilde{L}_{11} & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} \end{pmatrix}$$

with unit diagonals and a nonsingular upper triangular matrix

$$\tilde{U} = \begin{pmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ 0 & \tilde{U}_{22} \end{pmatrix}$$

such that

i) the entries of $\tilde{L}_{11}$ and $\tilde{L}_{21}$ are polynomials in T^* over $\underline{K}$ of degree at most 1.

ii) $\tilde{U}_{11} \in M(\underline{K})$ and the diagonal entries of $\tilde{U}_{22}$ are algebraically independent over $\underline{K}$.

This procedure is applied to the iterative solution of a system of linear/non-linear equations $f(x)=0$ in real unknown variables x, as follows. Let us suppose that a sequence of approximate solutions are computed by means of the Newton method, which would involve the solution of $J(x) \, \Delta x = f(x)$ for Δx through the LU-decomposition of $J(x)$, where $J(x)$ is the Jacobian matrix.

Since the non-constant derivatives of $f(x)$ may vary in value at each iteration, we regard them as being algebraically independent, or in other words, denoting the non-linear part of $J(x)$ by $T(x)$, we express $J(x)$ in the form (1):

$$J(x) = Q + T(x)$$

with $\underline{K}=\underline{Q}$ or $\underline{K}=\underline{R}$ (the field of real numbers). Furthermore we assume that $\det J(x) \in \underline{Q}^*$ or $\underline{R}^*$.

As the Theorem guarantees, we can obtain the LU-decomposition of $J(x)$:

$$J(x) = L(x)\ U$$

with

$$L(x) = (I + T(x)\ Q^{-1})\ M$$

$$= M + T(x)\ U^{-1},$$

where $Q = M\ U$, as above, and the permutation matrices are suppressed for simplicity. Since M and U do not depend on x, they can be computed before the iteration process starts. At each iteration step, only the L-factor $L(x)$ of $J(x)$ is to be computed. Note that U^{-1} on the right-hand side of $L(x)$ does not cost much since U is triangular. As pointed out in Remark 1 in the previous section, we may alternatively adopt the LU-decomposition $J(x) = L\ U(x)$ with the L-factor being independent of x.

Acknowledgement.

The author expresses his gratitude to Professor Masao Iri of the University of Tokyo for his penetrating suggestions. The author is supported by The Sakkokai Foundation of Japan.

References

[1] Murota, K., and Iri, M.: Combinatorial Approach to the Solvability of a System of Equations (in Japanese). *Proceedings of Fall Conference of the Operations Research Society of Japan*, D-II-1, pp.198-199, 1981.

[2] Iri, M.: The Maximum-Rank Minimum-Term-Rank Theorem for the Pivotal Transforms of a Matrix. *Linear Algebra and Its Applications*, Vol.2 (1969), pp.427 - 446.